Emission spectrum and Fourier transformations

Spectral expansion of the emission

Monochromatic light is an idealization. The light from real physical source can not be monochromatic as every process is finite. Thus said, the radiation can not be treated as ∞ harmonic function.

Limitation in time $\nrightarrow$ Monochromatic wave

However, any wave we can visualize as sum of monochromatic harmonic waves of different frequencies:

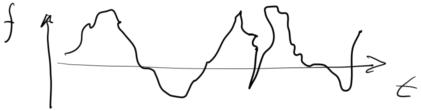
$$F(t) = \sum_{i}^{n} f(\omega_i)$$

This directly follows from superposition principle.

Here, superposition principle allows to explore complox e/m wave as a sum of simple monochromatic harmonic waves.

However, mathematically such procedure can be expressed as Fourier transformation.

1) Let's take periodic function



$$f(t) = f(t+T)$$

We know, that every periodic function we can present as sum of ∞ harmonic functions. (Peter Dirichlet 1829)

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos \omega_n t + \beta_n \sin \omega_n t)$$

$$\omega_n = \frac{2\pi}{T} n \longrightarrow$$ frequency that is even to $\frac{2\pi}{T}$, i.e. main frequency

$$\alpha_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t)\,dt$$

$$\alpha_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \omega_n t\,dt$$

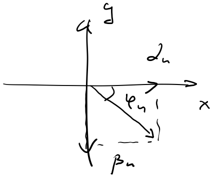
$$\beta_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \omega_n t\,dt$$

Since

$$\alpha_n \cos \omega_n t + \beta_n \sin \omega_n t =$$

$$= \alpha_n \cos \omega_n t + \beta_n \cos\left(\omega_n t - \frac{\pi}{2}\right)$$

This is in fact two vectors:



$$f(\omega_n) = \sqrt{\alpha_n^2 + \beta_n^2}$$

$$\varphi_n = tg\, \frac{\beta_n}{\alpha_n}$$

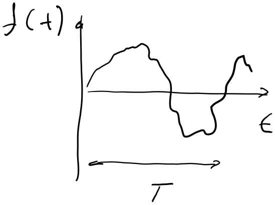
Then Fourier expansion is

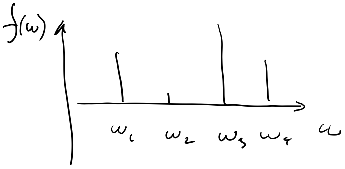
$$\boxed{f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} f(\omega_n) \cos(\omega_n t - \varphi_n)}$$

$f(\omega_n)$ - amplitude spectrum of $f(t)$

φ_n - phase spectrum of $f(t)$

Hence we have:





time language

spectral language

What does it mean?

It means that spectrum of periodic function is descrete.

Demonstration

Sound spectrum

Now let's look at the case of non-periodic function.

$$f(t) \text{ with } T \to \infty$$

First and foremost, this function has to be absolutely integrable:

$$\int_{-\infty}^{+\infty} |f(t)| \, dt < \infty$$

We can write:

$$f(t) = \frac{\alpha_0}{2} + \left(\frac{T}{2\pi}\right) \sum_{n=1}^{\infty} \left[\alpha_n \cos \omega_n t + \beta_n \sin \omega_n t\right] \Delta\omega$$

devide by $\Delta\omega$

multiply by $\Delta\omega$

$$\Delta\omega = \frac{2\pi}{T}$$

When $T \to \infty$, $\Delta\omega \to d\omega$

Hence

$$\lim_{T\to\infty} \frac{\alpha_0}{2} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt = 0$$

It is because $\int_{-T/2}^{T/2} f(t)\,dt$ is a limited number, but $T \to \infty$. This is why it is important $f(t)$ is integrable.

$$\lim_{T\to\infty} \frac{T}{2}\alpha_n = \lim_{T\to\infty} \frac{T}{2}\cdot\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\omega_n t\,dt =$$

$$= \int_{-\infty}^{+\infty} f(t)\cos\omega t\,dt = a(\omega)$$

$$a(\omega) = a(-\omega)$$

even function

$$\lim_{T\to\infty} \frac{T}{2}\beta_n = \lim_{T\to\infty} \frac{T}{2}\cdot\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\omega_n t\,dt =$$

$$= \int_{-\infty}^{+\infty} f(t)\sin\omega t\,dt = \beta(\omega)$$

$$\beta(\omega) = -\beta(-\omega)$$

odd function

Thus we have:

$$f(t) = \frac{1}{\pi}\int^{\infty}\left[a(\omega)\cos\omega t + \beta(\omega)\sin\omega t\right]d\omega$$

$\cup$

This is Fourier integral.

Let's introduce the function:

$$f(\omega) = a(\omega) - i\, b(\omega)$$

We also need to remember Euler formula:

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

Thus we will have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[a(\omega) - i\, b(\omega)\right]\cdot$$

$$\cdot\left[\cos\omega t + i\sin\omega t\right]d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Big(\underbrace{a(\omega)\cos\omega t}_{\text{even}} + \underbrace{b(\omega)\sin\omega t}_{\text{even}}\Big)d\omega + \frac{i}{2\pi}\int_{-\infty}^{+\infty}\Big[\underbrace{\cancel{a(\omega)\sin\omega t}}_{\text{odd}} -$$

$$- \underbrace{\cancelto{0}{b(\omega)\cos\omega t}}_{\text{odd}}\Big]d\omega = \breve{f}(t)$$

Thus

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)\, e^{i\omega t}\, d\omega}$$

At the same time:

$$\begin{cases} a(\omega) = \int\limits_{-\infty}^{+\infty} f(t)\cos\omega t\,dt \\ b(\omega) = \int\limits_{-\infty}^{+\infty} f(t)\sin\omega t\,dt \end{cases}$$

$$f(\omega) = \int\limits_{-\infty}^{+\infty} f(t)\cos\omega t\,dt - i\int\limits_{-\infty}^{+\infty} f(t)\sin\omega t\,dt =$$

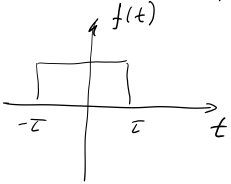
$$= \int\limits_{-\infty}^{+\infty} f(t)\,e^{-i\omega t}\,dt$$

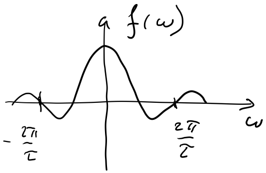
$$\boxed{f(\omega) = \int\limits_{-\infty}^{+\infty} f(t)\,e^{-i\omega t}\,dt}$$

Now, let's explore couple of examples:

① Rectangular pulse

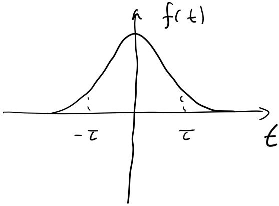
$$f(t) = \begin{cases} f_0, & |t| \le \tau \\ 0, & |t| > \tau \end{cases} \qquad f(\omega) = 2f_0 \operatorname{sinc} \omega\tau$$

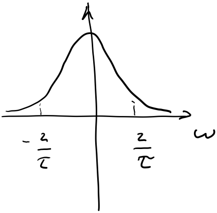




② Gauss pulse

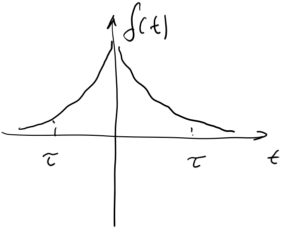
$$f(t) = f_0\,e^{-\frac{t^2}{\tau^2}} \qquad f(\omega) = \sqrt{\pi}\,f_0\,\tau\,e^{-\frac{\omega^2\tau^2}{4}}$$

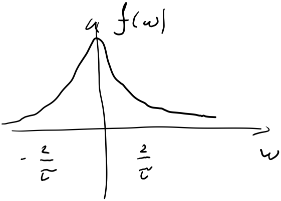




③ Exponential pulse

$$f(t) = f_0 e^{-\frac{|t|}{\tau}} \qquad f(\omega) = \frac{2\tau f_0}{1+\omega^2\tau^2}$$





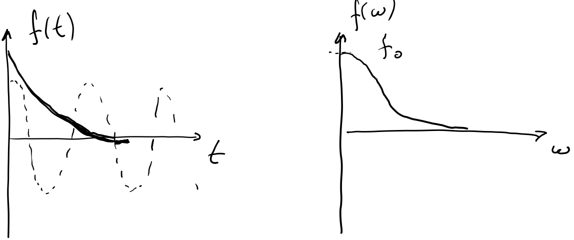
Lorentz

If one noticed, there is distinct connection between $\Delta\omega$ and Δt

$$\Delta\omega \sim \frac{1}{\Delta t}$$

This very much resonates with Heisenberg uncertainty principle, discovered 150 years later.

But if we want to look at it in details, let's take an example of laser pulse.

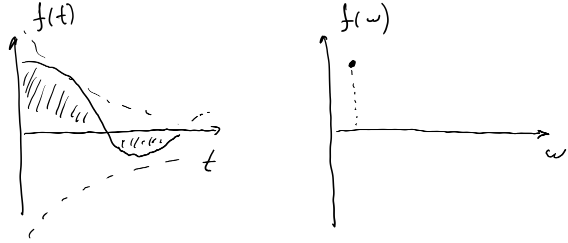


To understand it, let's take simplified Fourier transformation

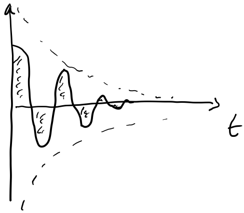
$$f(\omega) = \int_0^\infty f(t) \cos \omega t \, dt$$

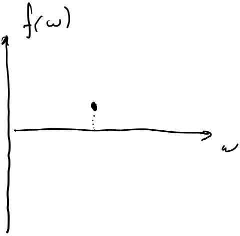
Basically, we need to look into integrated area, i.e. how cos ωt will fit under pulsed curve for different frequencies.

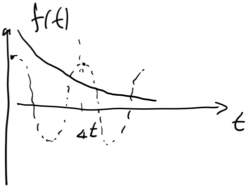
1)

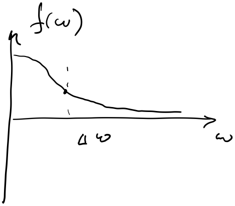


2)









$$T = \frac{2\pi}{\omega}$$

$$\begin{cases} f(\omega) \sim f(\omega = 0) & \omega \ll \dfrac{2\pi}{\Delta t} \\ f(\omega) \sim 0 & \omega \gg \dfrac{2\pi}{\Delta t} \end{cases}$$

$$\boxed{\Delta \omega = \frac{2\pi}{\Delta t}}$$

Now, all these allow us to talk about energy density spectrum.

$$W \sim \int_{-\infty}^{+\infty} E^2(t)\, dt = \int_{-\infty}^{+\infty} E(t) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} E(\omega)\, e^{i\omega t} d\omega \right] dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(\omega)\left[\int_{-\infty}^{+\infty} E(t)\, e^{i\omega t}\, dt\right] d\omega =$$

we exchanged integration

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(\omega)\, E(-\omega)\, d\omega = \frac{1}{2\pi}\int_{0}^{\infty} 2|E(\omega)|^2 d\omega$$

What does it mean?

Total energy of non-monochromatic wave is the integral of energy density on all the frequencies.